

Euler-Like Vector Fields and Manifolds with Filtered Structure

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Abstract

The first purpose of this note is to comment on a recent article of Bursztyn, Lima and Meinrenken, in which it is proved that if M is a smooth submanifold of a manifold V , then there is a bijection between germs of tubular neighborhoods of M and germs of “Euler-like” vector fields on V . We shall explain how to approach this bijection through the deformation to the normal cone that is associated to the embedding of M into V . The second purpose is to study generalizations to smooth manifolds equipped with Lie filtrations. Following in the footsteps of several others, we shall define a deformation to the normal cone that is appropriate to this context, and relate it to Euler-like vector fields and tubular neighborhood embeddings.

1 Introduction

The *Euler vector field* on a finite-dimensional real vector space V is the infinitesimal generator of the scalar multiplication flow. Thus if f is a smooth function on V , then its derivative in the direction of the Euler vector field is

$$(1.1) \quad E(f)(v) = \left. \frac{d}{dt} \right|_{t=0} f(e^t v).$$

If x_1, \dots, x_q is any linear coordinate system on V (in other words, a basis for the dual vector space V^*), then

$$(1.2) \quad E = x_1 \frac{\partial}{\partial x_1} + \dots + x_n \frac{\partial}{\partial x_n}.$$

The Euler vector field is also characterized by the property that if f is a smooth homogeneous function on V of degree q , then

$$(1.3) \quad E(f) = q \cdot f.$$

The concept of Euler vector field extends easily to vector bundles: if V is the total space of a smooth, real vector bundle over a smooth manifold M , then the *Euler vector field* on V is given by the formula (1.1) above, or equivalently by the obvious variations of (1.2) or (1.3). On each fiber, the Euler vector field of the bundle restricts to the Euler vector field of the fiber.

This note is about the following extension of the concept of Euler-vector field to the non-linear context:

1.1 Definition (See [BLM16, Definition 2.5]¹). If M is a smooth embedded submanifold of a smooth manifold V , then an *Euler-like vector field* for the embedding of M into V is a vector field E on V with the property that if f is a smooth function on V that vanishes on M to order $q \geq 1$, then

$$E(f) = q \cdot f + r,$$

where the remainder r is a smooth function that vanishes to order $q+1$ or higher (recall that smooth function f on V *vanishes to order* $q \geq 1$ *on* M if Df vanishes on M for every linear differential operator D on V of order $q-1$ or less).

If V is the total space of a vector bundle over M , then the Euler vector field on V is Euler-like for the embedding of M into V as the zero section. More generally, recall that a *tubular neighborhood* of M in V is a diffeomorphism from an open neighborhood of the zero section in the total space of the normal bundle

$$N_V M = TV|_M / TM$$

to an open neighborhood of M in V such that:

(1.1) the diffeomorphism is the identity on M (where M is embedded in the normal bundle as the zero section), and

(1.2) the differential of the diffeomorphism, restricted to vertical tangent vectors, induces the identity map from $N_V M$ to itself.

If E is the Euler vector field on the normal bundle, then any tubular neighborhood embedding carries E to an Euler-like vector field defined in a neighborhood of M in V . Let us call this the Euler-like vector field *associated* to the tubular neighborhood embedding.

Our first purpose is to comment on the following attractive result:

¹In [BLM16] it is required that Euler-like vector fields be complete. That is not necessary for our purposes and does not affect the results below, which concern germs of Euler-like vector fields near M .

1.2 Theorem (See [BLM16, Proposition 2.6]). *The correspondence that associates to a tubular neighborhood embedding its associated Euler-like vector field determines a bijection from germs of tubular neighborhoods to germs of Euler vector fields.*

The theorem has a number of applications, mostly stemming from the fact that any affine combination, with C^∞ -function coefficients, of Euler-like vector fields is again an Euler-like vector field. To give a very easy example, there exist equivariant tubular neighborhood embeddings for compact group actions since invariant Euler-like vector fields be constructed by partitions of unity and averaging. We refer to the article [BLM16] for more substantial results.

We shall examine Theorem 1.2 from the perspective of the *deformation to the normal cone* associated to the embedding of M into V , which in this paper we shall simply call the *deformation space* associated to the embedding. Among other things, the deformation space $N_V M$ is a smooth manifold that is equipped with a submersion onto \mathbb{R} . The fibers of this submersion over all nonzero $x \in \mathbb{R}$ are copies of V , while the fiber over $x = 0$ is the normal bundle for the embedding of M into V . So the deformation space may be described, as a set, as a disjoint union

$$N_V M = N_V M \times \{0\} \sqcup \bigsqcup_{\lambda \in \mathbb{R}^\times} V \times \{\lambda\}.$$

See Section 3 for further details, including, most importantly, a review of the smooth manifold structure on $N_V M$.

The deformation space can often serve as a stand-in for a tubular neighborhood embedding. It has the advantages of being canonical and of existing in contexts where tubular neighborhood embeddings need not exist (for example the category of complex manifolds).

If E is an Euler-like vector field on M , then there is an associated vector field \mathbb{E} on $N_V M$ that is vertical for the submersion to \mathbb{R} , restricts to a copy of E on each fiber $V \times \{\lambda\}$, and restricts to the Euler vector field on the zero fiber $N_V M \times \{0\}$.

There is also a canonical vector field \mathbb{A} on the deformation space that restricts to $\lambda \cdot \partial/\partial\lambda$ on the open set

$$V \times \mathbb{R}^\times = N_V M|_{\mathbb{R}^\times}$$

(moreover \mathbb{A} is vertical on the fiber over $0 \in \mathbb{R}$, and is the negative of the Euler vector field there). The formula

$$\lambda \cdot T = \mathbb{A} + \mathbb{E}$$

defines a third vector field \mathbb{T} on the deformation space. The time t flow map associated to the vector field \mathbb{T} sends the fiber of the deformation space over $\lambda=0$ to the fiber over $\lambda=t$, and its differential in the vertical direction is t times the identity (compare condition (1.2) above). The $t=1$ map is defined in a neighborhood of the zero section in the normal bundle, and is a tubular neighborhood embedding. So we have associated a tubular neighborhood embedding to the Euler vector field E , which is the main issue in proving Theorem 1.2.

Our approach throughout will be algebraic, treating vector fields very explicitly as derivations of algebras of smooth functions, and so on. Indeed we shall follow the algebraic-geometric approach and *define* the deformation space $N_V M$ as the spectrum of a suitable algebra (namely the Rees algebra associated to the filtration of smooth functions on V by order of vanishing on M). We have adopted this point of view because it fits very well with our second purpose, which is to study deformation spaces in the context of *filtered manifolds*.

A filtered manifold is a smooth manifold that is equipped with an increasing filtration on its tangent bundle which is compatible with Lie brackets of vector fields; see Definition 5.1 for details. The definition arose from problems in partial and pseudodifferential operator theory. More recently, filtered manifolds have received attention in noncommutative geometry thanks to work in index theory by Connes and Moscovici [CM95], Ponge [Pon00, Pon06] and Van Erp [Erp05, Erp10a, Erp10b].

A recurring theme in the theory of filtered manifolds is the importance of a family of unipotent “osculating groups” parametrized by the points of a filtered manifold. Our main observation (which is very simple) is that the osculating groups emerge very naturally from the algebraic approach we are taking this paper; see Section 5. As an aside, the algebraic approach also offers a new perspective on the construction of preferred coordinate systems on filtered manifolds; see Remark 5.23 for a discussion of this issue.

From these starting points it is not difficult to define counterparts in the filtered manifold context of the normal bundle, the deformation space, Euler-like vector fields, and so on. This we shall do in the remaining sections of the paper, where the main theorems are verbatim copies of our previous results for ordinary manifolds. But once again, there are interesting connections between our approach and the study of special coordinate systems on filtered manifolds—see for instance Remark 8.9.

2 Smooth Manifolds From Algebras

In this section we shall give some elementary algebraic definitions that we shall use throughout the paper, give criteria guaranteeing that the spectrum of an algebra carries a smooth manifold structure, and compare derivations on algebras to vector fields on spectra in the manifold case.

2.1 Definition. Let A be an associative and commutative² algebra with a multiplicative identity over the field of real numbers. A *character* of A is a nonzero algebra homomorphism

$$\varphi: A \longrightarrow \mathbb{R}.$$

The *spectrum* of A is the set of all characters. We equip it with the topology of pointwise convergence, that is, the topology having the fewest open sets so that the evaluation maps

$$\widehat{a}: \varphi \longmapsto \varphi(a)$$

are continuous functions on the spectrum, for every $a \in A$.

2.2 Definition. Let A be as above. Denote by \mathcal{S} the following sheaf of continuous, real-valued functions on the spectrum of A : if Ω is an open set in the spectrum, then a function f belongs to $\mathcal{S}(\Omega)$ if and only if for every character φ in Ω there are elements a_1, \dots, a_k in A (for some k) and a smooth, real-valued function g on \mathbb{R}^k such that

$$f = g(\widehat{a_1}, \dots, \widehat{a_k})$$

in some neighborhood of φ .

2.3 Definition. Let Ω be an open subset of $\text{Spectrum}(A)$ and let a_1, \dots, a_n be elements of A . We shall say that these elements *smoothly generate* $\mathcal{S}(\Omega)$ if for every $f \in \mathcal{S}(\Omega)$ there is a smooth function g on \mathbb{R}^n such that

$$f = g(\widehat{a_1}, \dots, \widehat{a_n})|_{\Omega}.$$

2.4 Lemma. Let A be a commutative algebra. The spectrum of A is a smooth manifold, with \mathcal{S} the sheaf of smooth functions, if and only if for every φ in the spectrum there is an open neighborhood Λ of φ , and there are elements a_1, \dots, a_n in A , such that

²It is not necessary to assume commutativity, but the definitions that follow are not very interesting in the noncommutative case.

(i) the elements a_1, \dots, a_n smoothly generate $\mathcal{S}(\Lambda)$, and

(ii) the map

$$(\widehat{a}_1, \dots, \widehat{a}_n): \text{Spectrum}(A) \longrightarrow \mathbb{R}^n$$

is a homeomorphism from Λ to an open set in \mathbb{R}^n . \square

For the rest of this section we shall assume that the spectrum of A is indeed a smooth manifold, with \mathcal{S} the sheaf of smooth functions.

2.5 Definition. Let X be a derivation of the algebra A , and let Ω be an open subset of the spectrum of A . We shall say that X extends to a vector field \widehat{X} on Ω if the diagram

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto \widehat{a}} & C^\infty(\Omega) \\ X \downarrow & & \downarrow \widehat{X} \\ A & \xrightarrow{a \mapsto \widehat{a}} & C^\infty(\Omega) \end{array}$$

commutes.

An obvious necessary condition for X to be extendable to a vector field on Ω is that if $\Lambda \subseteq \Omega$ is any open subset, if

$$a, a_1, \dots, a_k \in A,$$

if g is a smooth function of k variables, and if

$$(2.1) \quad \widehat{a}|_\Lambda = g(\widehat{a}_1, \dots, \widehat{a}_k)|_\Lambda,$$

then

$$(2.2) \quad \widehat{X(a)}|_\Lambda = \sum_{i=1}^k \widehat{X(a_i)}|_\Lambda \cdot g_i(\widehat{a}_1, \dots, \widehat{a}_k)|_\Lambda$$

where g_i denotes the i 'th partial derivative of g .

2.6 Definition. If Ω is an open subset of the spectrum, then we shall say that a derivation X of A is *smooth over* Ω if (2.2) holds for every open subset $\Lambda \subseteq \Omega$, all $a, a_1, \dots, a_k \in A$, and all g as in (2.1).

2.7 Lemma. *If the spectrum of A is a smooth manifold, then every derivation of A that is smooth over an open subset Ω of the spectrum of A extends to a unique vector field on Ω .*

Proof. By Lemma 2.4, around every point of the spectrum there is a neighborhood Λ , and elements a_1, \dots, a_n of A , so that $\widehat{a_1}, \dots, \widehat{a_n}$ are coordinate functions on Λ . Since any vector field on Λ is completely determined by its action on a system of coordinate functions, we see that there is at most one vector field on Ω that extends X .

As for existence, given local coordinates of the type $\widehat{a_1}, \dots, \widehat{a_n}$ on some open subset Λ of Ω , we can define \widehat{X} on Λ by

$$\widehat{X}(g(\widehat{a_1}, \dots, \widehat{a_n})) = \sum_{i=1}^n \widehat{X(a_i)} \cdot g_i(\widehat{a_1}, \dots, \widehat{a_n}).$$

This is a vector field, it extends X on Λ by (2.2), and it is independent of the choice of local coordinates, again by (2.2). So we obtain a global extension over Ω , as required. \square

In our calculations it will be helpful to observe the following fact:

2.8 Lemma. *Let Ω be an open subset of the spectrum of A whose complement has empty interior. Every derivation of A that is smooth over Ω is smooth over the full spectrum.*

Proof. Let Λ be an open subset of the spectrum. By hypothesis, any identity of smooth functions (2.1) over Λ leads to an identity of the type (2.2) over $\Lambda \cap \Omega$. But $\Lambda \cap \Omega$ is dense in Λ , so the identity (2.2) holds over Λ . \square

3 Deformation Space for Smooth Manifolds

Let V be a smooth manifold and let M be a smooth, embedded submanifold (both without boundary, as will always be the case in this paper). In this section we shall review the construction of the *deformation to the normal cone*, or *deformation space*, associated to the inclusion of M into V . See [Ful98, Chapter 5] for the standard treatment in algebraic geometry and see for example [Hig10] for the C^∞ -version.

We shall emphasize the algebraic aspects of the construction. These play only a modest role for ordinary manifolds, but they will be helpful when we consider filtered manifolds later on.

Here is a summary of what we shall do. The *deformation space* associated to the embedding of M into V may be described, as a set, as a disjoint union

$$(3.1) \quad \mathbb{N}_V M = \mathbb{N}_V M \times \{0\} \sqcup \bigsqcup_{\lambda \in \mathbb{R}^\times} V \times \{\lambda\},$$

as we noted in the introduction. It is given the weakest topology so that the obvious maps to \mathbb{R} and to V are continuous, and so that, in addition, for every smooth function α on V that vanishes on M , the function

$$\begin{aligned}(X_m, 0) &\longmapsto X_m(\alpha) \\ (v, \lambda) &\longmapsto \lambda^{-1} \alpha(v)\end{aligned}$$

is also continuous. Here X_m is a normal vector at $m \in M$, that is, a vector in the quotient space $T_m V / T_m M$. The value $X_m(\alpha)$ is well defined because α vanishes on M . We shall prove that the deformation space carries a smooth manifold structure so that all the functions above are smooth, and these functions generate all smooth functions in the natural sense (compare Definitions 2.2 and 2.6).

Now we proceed with the details.

3.1 Definition. Denote by $A(V, M)$ the \mathbb{R} -algebra of all Laurent polynomials

$$f(t) = \sum_{q \in \mathbb{Z}} \alpha_q t^{-q}$$

whose coefficients are smooth, real-valued functions on V that satisfy the condition

$$q > 0 \quad \Rightarrow \quad \alpha_q \text{ vanishes to order } q \text{ on } M$$

(this is indeed an algebra, because if α_p vanishes to order p on M , and α_q vanishes to order q on M , then the pointwise product $\alpha_p \alpha_q$ vanishes to order $p + q$). The *deformation space* $\mathbb{N}_V M$ is the spectrum of $A(V, M)$.

Our first objective is to identify $\mathbb{N}_V M$, defined as a spectrum, with (3.1). The inclusion into $A(V, M)$ of the polynomials in t with real coefficients induces a continuous map

$$(3.2) \quad \mathbb{N}_V M \longrightarrow \mathbb{R},$$

and we shall compute the fibers over each $\lambda \in \mathbb{R}$. These are the spectra of the following algebras:

3.2 Definition. For $\lambda \in \mathbb{R}$ denote by $A_\lambda(V, M)$ the quotient of $A(V, M)$ by the ideal generated by $t - \lambda$.

3.3 Lemma. *If $\lambda \in \mathbb{R}$ is nonzero, then $A_\lambda(V, M)$ is isomorphic to $C^\infty(V)$ via evaluation of Laurent polynomials at $t = \lambda$.*

Proof. If the element $\sum a_q t^{-q}$ lies in the kernel of evaluation at λ , then

$$\sum a_q t^{-q} = (t - \lambda) \cdot \sum_q \left(\sum_{j \geq 0} a_{q-j} \lambda^j \right) t^{-q-1},$$

and the right-most Laurent polynomial lies in $A(V, M)$, as required. \square

To handle the case where $\lambda = 0$ we need some notation.

3.4 Definition. For each integer $q > 0$ denote by $I_q(V, M)$ the ideal of smooth functions on V that vanish to order q on M . Set $I_0(V, M) = C^\infty(V)$.

The spaces $I_q(V, M)$ form a decreasing sequence of ideals in the algebra of smooth functions on V , and we can form the associated graded algebra

$$(3.3) \quad \bigoplus_{q \geq 0} I_q(V, M) / I_{q+1}(V, M).$$

If $a \in I_q(V, M)$, then we shall write

$$(3.4) \quad \langle a \rangle_q \in I_q(V, M) / I_{q+1}(V, M)$$

for the coset of $a \in I_q(V, M)$ in the degree q component of (3.3).

3.5 Lemma. *The algebra $A_0(V, M)$ is isomorphic to the associated graded algebra (3.3) via the map*

$$\sum_{q \in \mathbb{Z}} a_q t^{-q} \mapsto \sum_{q \geq 0} \langle a_q \rangle_q. \quad \square$$

It is now easy to compute the spectrum of $A_0(V, M)$. The degree zero part of $A_0(V, M)$ is $C^\infty(M)$, and each character of $A_0(V, M)$ restricts to evaluation at some point $m \in M$ on the degree zero part. The character therefore factors through the quotient algebra $A_{0,m}(V, M)$ by the ideal in $A_0(V, M)$ generated by the vanishing ideal of m in $C^\infty(M)$.

3.6 Lemma. *There is a unique isomorphism from $A_{0,m}(V, M)$ to the algebra of real-valued polynomial functions on the normal vector space $T_m V / T_m M$ for which*

$$\langle a \rangle_1 \mapsto [X_m \mapsto X_m(a)].$$

for every normal vector X_m and every smooth function a on V vanishing on M . The spectrum of $A_{0,m}(V, M)$ identifies in this way with $T_m V / T_m M$. \square

3.7 Remark. We shall prove a more general result in Theorem 6.8.

Returning to the deformation space, the above considerations identify the fibers of (3.2) with V when $\lambda \neq 0$, and with the normal bundle $N_V M$ when $\lambda = 0$. We obtain the description (3.1), as required. As for the topology on $N_V M$, since $A(V, M)$ is generated by:

- (a) the element $t \in A(V, M)$,
- (b) the functions $a \cdot t^0 \in A(V, M)$, where $a \in C^\infty(V)$, and
- (c) monomials $a \cdot t^{-1} \in A(V, M)$, where a vanishes on M .

we find that the topology on $N_V M$, viewed as a spectrum, agrees with the topology we described earlier.

3.8 Theorem. *The deformation space $N_V M$ is a smooth manifold.*

Proof. We shall use Lemma 2.4. The only nontrivial case is that of a character φ in the fiber over $\lambda = 0$, corresponding to a normal vector X_m . Introduce smooth functions x_1, \dots, x_n on V that are local coordinates in a neighborhood U of m in V , for which

$$M \cap U = \{u \in U : x_{k+1}(u) = \dots = x_n(u) = 0\}.$$

Now define $\Lambda \subseteq N_V M$ to be the open set consisting of those elements of the deformation space of the form (u, λ) for $u \in U$ and $\lambda \neq 0$, or $(X_u, 0)$ for $u \in M \cap U$. The elements

$$(3.5) \quad t, x_1, \dots, x_k, x_{k+1}t^{-1}, \dots, x_nt^{-1} \in A(V, M)$$

satisfy the hypotheses of Lemma 2.4; if $W \subseteq \mathbb{R}^n$ is the image of U under the coordinates $\{x_j\}$ on V , then the homeomorphic image of Λ under the functions (3.5) is the open set

$$\{(\lambda, x_1, \dots, x_n) : (x_1, \dots, x_k, \lambda x_{k+1}, \dots, \lambda x_n) \in W\}$$

in \mathbb{R}^{n+1} ; and the smooth generation statement in the lemma follows from the Taylor expansion for smooth functions on V . \square

4 Vector Fields on the Deformation Space

In this section we shall give a proof of Theorem 1.2 (the theorem of Bursztyn, Lima and Meinrenken) using vector fields on the deformation space

$\mathbb{N}_V M$. The original proof in [BLM16] is not difficult, but we think that the argument presented below offers an interesting new perspective.

The main task is to show that every Euler-like vector field E can be identified with the Euler vector field on the normal bundle under *some* tubular neighborhood embedding, because uniqueness of the germ of the tubular neighborhood embedding, given existence, is easy to establish. Indeed the composition of one embedding with the inverse any other is a diffeomorphism from one neighborhood of the zero section in the normal bundle to another that fixes the zero section, and, if the two embeddings are associated to the same Euler-like vector field, commutes with scalar multiplication. Any such diffeomorphism is fiber-preserving and linear on each fiber, and condition (1.2) ensures that it is the identity map.

4.1 Lemma. *The formula*

$$\alpha_s: \begin{cases} (v, \lambda) \mapsto (v, e^s \lambda) \\ (X, 0) \mapsto (e^{-s} X, 0) \end{cases}$$

defines a smooth action of the Lie group \mathbb{R} on the deformation space $\mathbb{N}_V M$.

Proof. This is easy to check directly in the local coordinates of Theorem 3.8. Fom the algebraic point of view, it suffices to note that the geometric flow is associated to the morphism

$$\alpha: A(V, M) \longrightarrow A(V, M) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}),$$

defined by the formula

$$\alpha: \sum a_q t^{-q} \mapsto \sum a_q t^{-q} \otimes e^{-tq}$$

(the tensor product here is the ordinary algebraic tensor product). □

4.2 Definition. We shall denote by \mathbb{A} the vector field on $\mathbb{N}_V M$ that generates the flow $\{\alpha_s\}$ above.

Note that \mathbb{A} restricts to the vector field $\lambda \cdot \partial/\partial \lambda$, on $V \times \mathbb{R}^\times$, while on the zero fiber of $\mathbb{N}_V M$ it agrees with the negative of the Euler vector field on the normal bundle.

4.3 Lemma. *Let E be an Euler-like vector field for the inclusion of M into V . The vector field*

$$\mathbb{T} = \lambda^{-1} E + \frac{\partial}{\partial \lambda}$$

on the open subset $V \times \mathbb{R}^\times \subseteq \mathbb{N}_V M$ extends to a (smooth) vector field on $\mathbb{N}_V M$ with

$$\lambda \cdot \mathbb{T} = \mathbb{A} + \mathbb{E}.$$

Proof. If \mathbb{E} is Euler-like, then the formula

$$(4.1) \quad \sum a_q t^{-q} \mapsto \sum (\mathbb{E}(a_q) - q a_q) t^{-(q+1)},$$

defines a derivation of $A(V, M)$. The derivation extends to the vector field \mathbb{T} over the open set $V \times \mathbb{R}^\times \subseteq \mathbb{N}_V M$, and since the complement of this open set has empty interior it follows from Lemma 2.8 that the derivation extends to a vector field on all of $\mathbb{N}_V M$. \square

Denote by $\{\tau_s\}$ the local flow on $\mathbb{N}_V M$ associated to the vector field \mathbb{T} in Lemma 4.3. Recall that the maps τ_s assemble into a smooth map

$$\tau: \mathbb{R} \times \mathbb{N}_V M \longrightarrow \mathbb{N}_V M$$

that is defined on some neighborhood of $\{0\} \times \mathbb{N}_V M$ in $\mathbb{R} \times \mathbb{N}_V M$, such that

$$(4.2) \quad \mathbb{T}(f)(w) = \left. \frac{d}{ds} \right|_{s=0} f(\tau_s(w))$$

for all smooth functions f on $\mathbb{N}_V M$ and all $w \in \mathbb{N}_V M$, and

$$(4.3) \quad \tau_{s+t}(w) = \tau_s(\tau_t(w))$$

in a neighborhood of $\{0\} \times \{0\} \times \mathbb{N}_V M$ in $\mathbb{R} \times \mathbb{R} \times \mathbb{N}_V M$. In all these formulas, we are writing $\tau_s(w) = \tau(s, w)$. If $s > 0$, then for $(X, 0) \in \mathbb{N}_V M$ we can write

$$\mathbb{N}_V M \ni (X, 0) \xrightarrow{\tau_s} (\varphi_s(X), s) \in \mathbb{N}_V M.$$

Observe first that φ_s maps the zero section in the normal bundle identically to $M \subseteq V$ because \mathbb{T} restricts to the vector field $\partial/\partial\lambda$ on the submanifold

$$M \times \mathbb{R} \subseteq \mathbb{N}_V M.$$

The second property of φ_s that we shall need is as follows:

4.4 Lemma. *If K is a compact subset of $\mathbb{N}_V M$ and $k > 0$, then there exists $\varepsilon > 0$ so that*

$$\varphi_{e^t s}(X) = \varphi_s(e^{-t} X)$$

for all $X \in K$, all $|t| < k$, and all $s \in (-\varepsilon, \varepsilon)$.

Proof. It follows directly from the definitions of \mathbb{T} and α_t that

$$(4.4) \quad e^t \mathbb{T}_{(v, \lambda)} = \alpha_{t*} \mathbb{T}_{(v, e^{-t} \lambda)},$$

and it follows from this that

$$(4.5) \quad \tau_{e^t s} = \alpha_t \circ \tau_s \circ \alpha_{-t}$$

since the left and right hand sides of (4.4) are the generators of the left and right hand flows in s in (4.5). The formula in the lemma follows by evaluating both sides on $(X, 0)$. \square

4.5 Theorem. *If E is an Euler-like vector field for the inclusion of M into V , then there is a tubular neighborhood diffeomorphism*

$$\Phi: N_V M \longrightarrow V$$

(defined on a neighborhood of the zero section) that carries the Euler vector field on the normal bundle to the germ of E near M .

Proof. Choose a neighborhood of the zero section in $N_V M$ and a smooth positive function $s(m) = e^{-t(m)}$ so that $\varphi_s(X_m)$ is defined for all $X_m \in U$ and all $|s| < 2s(m)$. Using Lemma 4.4, we find that the formula

$$\Phi(X_m) = \varphi_{e^{-t(m)}}(e^{t(m)} X_m)$$

defines a tubular neighborhood mapping on a neighborhood of the zero section. \square

5 Lie Filtrations and Unipotent Groups

In this section we shall review the definition of a *Lie filtration* on the tangent bundle of a smooth manifold, due to Melin [Mel82] and give an algebraic description of the unipotent *osculating groups* that are attached to the points of a filtered manifold.

5.1 Definition. Let V be a smooth manifold. A *Lie filtration* on the tangent bundle TV is an increasing sequence of smooth vector subbundles

$$H^1 \subseteq H^2 \subseteq \cdots \subseteq H^r = TV$$

with the property that if X and Y are vector fields on V , and also sections of H^p and H^q , respectively, then the Lie bracket $[X, Y]$ is a section of H^{p+q} (we set $H^{p+q} = TV$ if $p+q \geq r$). An *r -step filtered manifold* is a smooth manifold whose tangent bundle is equipped with a Lie filtration of length r , as above.

5.2 Remark. The concept of filtered manifold has arisen in a number of places beyond partial differential equations. See for instance the works [Mor93], [Bel96] and [ČS09].

We shall usually write (V, H) to make explicit reference to the Lie filtration. For simplicity we shall assume throughout that the bundles H^q in Definition 5.1 have constant rank, which of course they must have if V is connected.

5.3 Example. An ordinary smooth manifold is of course a 1-step filtered manifold. In the 1-step case the constructions in this and the next two sections will be identical with the constructions in Section 3.

5.4 Example. In the 2-step case the Lie bracket condition in Definition 5.1 is vacuous, so a 2-step filtered manifold is simply a smooth manifold together with a smooth vector subbundle of the tangent bundle (Beals and Greiner [BG88] coined the term *Heisenberg manifold* for the special case in which this bundle has codimension one in the tangent bundle). The calculations in this and the following sections are very easy in the 2-step case.

See [CP15] and [EY15] for further discussion and examples of the concept of filtered manifold.

For our purposes, the significant features of a filtered manifold (V, H) will be accessed through the algebra of linear partial differential operators on V , and in particular through an increasing filtration on differential operators that is determined by the Lie filtration on TV , as follows.

We begin with some generalities on differential operators, unrelated to Lie filtrations. If X_1, \dots, X_n is any local frame for the tangent bundle of a smooth manifold, then any linear partial differential operator D can be expressed in a unique way as a linear combination

$$(5.1) \quad D = \sum_{\alpha} f_{\alpha} X^{\alpha},$$

where

- (i) the sum is over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with nonnegative integer entries,
- (ii) $X^{\alpha} = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ (note that the order of X_1, \dots, X_n is fixed), and
- (iii) the coefficients f_{α} are smooth functions, all but finitely many of them zero.

5.5 Lemma. Let v be a point in a smooth manifold V , let $\{X_1, \dots, X_n\}$ be a local frame for TV , defined near v . If a linear differential operator D is expressed in terms of the frame as in (5.1), and if D vanishes at v in the sense that $(Df)(v) = 0$ for every smooth function f on V , then all the functions f_α vanish at v . \square

The following two definitions are taken from the work of Choi and Ponge [CP15, Section 2] (which in turn adapts terminology from [Bel96, Section 4]).

5.6 Definition. Let (V, H) be a r -step filtered manifold. A *local H-frame* for V is a local frame X_1, \dots, X_n for the tangent bundle such that for every $q = 1, \dots, r$, the vector fields

$$X_1, \dots, X_{\text{rank}(H^q)}$$

are sections of H^q , and so constitute a local frame for H^q .

5.7 Definition. The *weight sequence* of V is the sequence

$$(q_1, \dots, q_n) = (1, \dots, 1, 2, \dots, 2, \dots, r, \dots, r)$$

in which each integer q is repeated $\text{rank}(H^q) - \text{rank}(H^{q-1})$ times.

5.8 Remark. With this terminology, if $\{X_a\}$ is a local H -frame, then X_a is a section of the vector bundle H^{q_a} .

5.9 Definition ([Mel82, Section 3]). Let (V, H) be an r -step filtered manifold. Let D be a linear differential operator and let s be a nonnegative integer. We shall write

$$\text{order}_H(D) \leq s,$$

and say that the H -order of D is no more than s , at a point $v \in V$, if for some (or equivalently every) local H -frame X_1, \dots, X_n defined near v , the operator D can be expressed as a sum

$$(5.2) \quad D = \sum_{\alpha} f_{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n},$$

in such a way that

$$q_1 \alpha_1 + \cdots + q_n \alpha_n > s \quad \Rightarrow \quad f_{\alpha} = 0,$$

where $\{q_a\}$ is the weight sequence for (V, H) .

5.10 Example. In the 1-step case (see Example 5.3) this is of course the usual notion of order of a differential operator.

5.11 Definition. Let (V, H) be a filtered manifold and denote by $\mathcal{D}(V)$ the algebra of linear partial differential operators on V . We shall denote by

$$\mathcal{D}^s(V) \subseteq \mathcal{D}(V)$$

the linear space of all operators that are of H -order no more than s at every point of V .

It is evident that p and q are any nonnegative integers, then

$$\mathcal{D}^p(V) \cdot \mathcal{D}^q(V) \subseteq \mathcal{D}^{p+q}(V),$$

so the concept of H -order defines an increasing filtration on the algebra $\mathcal{D}(V)$. If X is a vector field on V and a section of H^q then X has H -order no more than q , and if it has order no more than $q-1$, then in fact it is a section of H^{q-1} .

The notion of H -order on differential operators leads to the following notion of order of vanishing of a function at a point in a filtered manifold:

5.12 Definition. Let V be a filtered manifold V and let v be a point in V . Let q be a positive integer. A smooth function f on V *vanishes to H -order q at v* if the function Df vanishes at v for every differential operator D of H -order $q-1$ or less. We shall denote by

$$I_q(V, v) \subseteq C^\infty(V)$$

the space of smooth, real-valued functions on V that vanish to H -order q . For convenience we shall also write $I_0(V, v) = C^\infty(V)$.

It is clear that the spaces $I_q(V, v)$ decrease as q increases. In addition

$$I_p(V, v) \cdot I_q(V, v) \subseteq I_{p+q}(V, v).$$

for all $p, q \geq 0$. So we obtain a decreasing filtration of the algebra $C^\infty(V)$.

5.13 Definition. Let v be a point in a filtered manifold (V, H) . Denote by $A_0(V, v)$ the associated graded algebra

$$A_0(V, v) = \bigoplus_{q \geq 0} I_q(V, v) / I_{q+1}(V, v).$$

In the context of ordinary manifolds this is the algebra of polynomial functions on the tangent space $T_v V$. Our objective in the remainder of this section is to show that $A_0(V, v)$ is the algebra of polynomial functions on a real unipotent group \mathcal{H}_v attached to the Lie filtration and the point $v \in V$.

5.14 Definition. Let (V, H) be a filtered manifold and let $v \in V$. Denote by \mathfrak{h}_v the direct sum

$$\mathfrak{h}_v = \bigoplus_{q=1}^r H_v^q / H_v^{q-1}.$$

Equip \mathfrak{h}_v with a graded Lie algebra structure, as follows. Given elements $\langle X_v \rangle_p$ and $\langle Y_v \rangle_q$ in degrees p and q , represented by tangent vectors $X_v \in H_v^p$ and $Y_v \in H_v^q$, extend both to sections of H^p and H^q and define

$$[\langle X_v \rangle_p, \langle Y_v \rangle_q] = \langle [X, Y]_v \rangle_{p+q}.$$

For further details, and examples, see [Mel82], [CP15] or [EY15].

5.15 Lemma. *The graded Lie algebra \mathfrak{h}_v acts as derivations on the graded algebra $A_0(V, v)$ via the formula*

$$\delta_{\langle X_v \rangle_p} : \sum_{q \geq 0} \langle a_q \rangle_q \longmapsto \sum_{q \geq p} \langle X(a_q) \rangle_{q-p},$$

where X_v is extended to a section X of H^p , as in Definition 5.14 (and where the angle-bracket notation $\langle a \rangle_q$ is as in (3.4)). \square

5.16 Definition. We shall denote by \mathcal{H}_v the unipotent group with Lie algebra \mathfrak{h}_v . This is the *osculating group* attached to the point v . Denote by $A(\mathcal{H}_v)$ the algebra of real-valued polynomial functions on \mathcal{H}_v .

5.17 Remark. In the present context *unipotent group* means the same thing as *simply connected nilpotent Lie group*, while $A(\mathcal{H}_v)$ is the algebra of functions on the group that correspond to polynomial functions on the Lie algebra \mathfrak{h}_v under the exponential map

$$\exp : \mathfrak{h}_v \longrightarrow \mathcal{H}_v,$$

which, we recall, is a diffeomorphism. See for example [Hoc81, Chapter XVI, Section 4] a more algebraic approach to the construction of \mathcal{H}_v .

Now if A is an algebra that is equipped with a locally finite-dimensional and locally nilpotent action of a finite-dimensional real nilpotent Lie algebra \mathfrak{h} by derivations, then the action of \mathfrak{h} exponentiates to an action of the

associated unipotent group \mathcal{H} by algebra automorphisms. And if ε is any character ε of A , then there is an *orbit homomorphism*³

$$(5.3) \quad A \longrightarrow A(\mathcal{H})$$

into the algebra of real-valued polynomial functions on the associated unipotent group that is defined by means of the formula

$$(5.4) \quad a \longmapsto [h \mapsto \varepsilon(h^{-1}(a))] \quad (a \in A, \quad h \in \mathcal{H}).$$

It is an \mathcal{H} -equivariant algebra homomorphism if we let \mathcal{H} act on $A(\mathcal{H})$ by the left regular representation.

5.18 Definition. We shall call the character

$$A_0(V, v) \ni \sum \langle a_q \rangle_q \xrightarrow{\varepsilon} a_0(v) \in \mathbb{R}$$

the *counit* of $A_0(V, v)$.

We shall prove the following result.

5.19 Theorem. *Let (V, H) be a filtered manifold, and let v be a point in V . The orbit homomorphism*

$$A_0(V, v) \longrightarrow A(\mathcal{H}_v)$$

associated to the counit of $A_0(V, v)$ is an \mathcal{H}_v -equivariant algebra isomorphism.

5.20 Remark. The orbit homomorphism in the theorem is the *unique* \mathcal{H}_v -equivariant homomorphism for which the composition

$$A_0(V, v) \longrightarrow A(\mathcal{H}_v) \xrightarrow{\text{eval. at } e} \mathbb{R}$$

is the counit of $A_0(V, v)$.

5.21 Lemma. *Let V be a filtered manifold of rank r , and let v be a point in V . Let $\{q_1, \dots, q_n\}$ be the weight sequence for (V, H) and let $\{X_a\}$ be a local H -frame, defined near v . There are local coordinates $\{x_a\}$ defined near v such that*

- (i) *each x_a vanishes at v to H -order q_a , and*
- (ii) *$X_a(x_b) = \delta_{ab}$ at the point v , for all $a, b = 1, \dots, n$.*

³It is dual to the orbit map $\mathcal{H} \rightarrow \text{Spectrum}(A)$ given by $h \mapsto h(\varepsilon)$.

Proof. Define a linear transformaton from $\mathcal{D}(V)$ into the vector space dual of $C^\infty(V)$ by means of the formula

$$D \longmapsto [f \mapsto (Df)(v)].$$

It induces a linear map

$$(5.5) \quad \mathcal{D}^r(V) \longrightarrow (C^\infty(V)/I_{r+1}(V, v))^*,$$

and let us observe here that the quotient $C^\infty(V)/I_{r+1}(V, v)$ is a *finite-dimensional* vector space.

It follows from Lemma 5.5 that the image under (5.5) of the set of monomial differential operators X^α of H-order no more than r is a linearly independent set. So by linear algebra there are functions $f_\beta \in C^\infty(V)$ with

$$(X^\alpha f_\beta)(v) = \delta_{\alpha\beta}$$

The members $\{x_a\}$ of this list of functions that correspond to the vector fields $\{X_a\}$ form a local coordinate system of the required type. \square

5.22 Remark. The coordinates provided by the lemma above are called *privileged coordinates* in [CP15, Definition 4.9] and [Bel96], and their existence is proved in [CP15, Proposition 4.13] and in [Bel96, Theorem 4.15]. Our argument is only slightly different.

Proof of Theorem 5.19. Equip the algebra $A(\mathcal{H}_v)$ with the decreasing filtration given by order of vanishing, in the ordinary sense unrelated to Lie filtrations, at $e \in \mathcal{H}_v$. The associated graded algebra is the symmetric algebra on its degree one part, which identifies with \mathfrak{h}^* .

The algebra $A_0(V, v)$ also carries a decreasing filtration, in which an element has order j or more if it can be represented as a sum $\sum \langle a_q \rangle_q$, with each a_q vanishing, also in the ordinary sense, to order j or more. The associated graded algebra is a symmetric algebra on the degree-one classes determined by the elements $\langle x_a \rangle_{q_a}$, where $\{x_a\}$ is any coordinate system as Lemma 5.21.

The filtrations of $A_0(V, v)$ and $A(\mathcal{H}_v)$ are compatible with one another under the map (5.3), and the generators $\langle x_a \rangle_{q_a}$ map to the dual basis elements

$$\langle X_{a,v} \rangle_{q_a}^* \in \mathfrak{h}_v^*,$$

with $\{X_a\}$ the local H-frame in Lemma 5.21. This proves the theorem. \square

5.23 Remark. Let $\{X_a\}$ be a local H-frame near $v \in V$, and let $\{x_a\}$ be an associated system of privileged coordinates, as in Lemma 5.21. The frame determines a basis $\{\langle X_{a,v} \rangle_{q_a}\}$ for the Lie algebra \mathfrak{h}_v and the local coordinates determine a local diffeomorphism

$$w \mapsto \sum_a x_a(w) \langle X_{a,v} \rangle_{q_a}$$

from V to \mathfrak{h}_v , and hence, by exponentiation, a local diffeomorphism

$$V \xrightarrow{\cong} \mathcal{H}_v.$$

This in turn induces an isomorphism of algebras

$$A(\mathcal{H}_v) \xrightarrow{\cong} A_0(V, v).$$

The algebra isomorphism depends on the choice of coordinate systems $\{x_a\}$, in general, and is *not* in general inverse to the canonical isomorphism of Theorem 5.19. Those coordinates for which the two isomorphisms *are* inverse to one another are called *Carnot coordinates* in [CP15].

6 Normal Spaces for Filtered Manifolds

In this section we shall construct the filtered manifold analogue of the normal bundle. Its fibers will be most naturally viewed as unipotent homogeneous spaces rather than as quotients of tangent vector spaces.

6.1 Definition. Let (V, H) be an r -step filtered manifold. An embedded submanifold $M \subseteq V$ is a *filtered submanifold* if the intersections

$$G^q = H^q|_M \cap TM \quad (q = 1, \dots, r)$$

are smooth vector subbundles of TM .

If M is a filtered submanifold of (V, H) , then the bundles G^q form a Lie filtration of TM , so that (M, G) is a filtered manifold in its own right.

6.2 Definition. Let (M, G) be a filtered submanifold of a filtered manifold (V, H) , and denote by $I_q(V, M)$ the ideal of smooth functions on V that vanish to H -order at least q on M . We shall denote by $A_0(V, M)$ the associated graded algebra

$$A_0(V, M) = \bigoplus_{q \geq 0} I_q(V, M) / I_{q+1}(V, M)$$

The *normal space* $N_V M$ is the spectrum of $A_0(V, M)$.

6.3 Theorem. *Let (M, G) be a filtered submanifold of a filtered manifold (V, H) . The normal space $N_V M$ is a smooth manifold in such a way that the sheaf of smooth functions is the sheaf \mathcal{S} from Definition 2.2.*

The proof is not difficult, but it requires some information about vector fields and local coordinates adapted to the inclusion of M into V .

6.4 Definition. Let (M, G) be a filtered submanifold of a filtered manifold (V, H) . A local (G, H) -frame for TV at a point of M is a local H -frame for V with the additional property that the vector fields in the frame that are tangent to M (upon restriction to M) form a local G -frame for M .

The vector fields in the local frame divide into two sets:

- (i) vector fields tangent to M upon restriction to M , which restrict to give a G -local frame for M , and
- (ii) vector fields not tangent to M .

We shall call the latter the *normal* vector fields in the local frame. The normal vector fields X_α for which $\alpha \leq \text{rank}(H^p)$ restrict to give a local frame for the quotient bundle $H^p|_M / G^p$.

6.5 Lemma. *Let (V, H) be an r -step filtered manifold with order sequence $\{q_\alpha\}$, and let (M, G) be a filtered submanifold of V . Let $\{X_\alpha\}$ be a local (G, H) -frame defined near a point $m \in M$. There are smooth functions z_c defined near m , one for each normal vector field Z_c in the frame, such that*

- (i) z_c vanishes on M to H -order q_c .
- (ii) $Z_c(z_d) = \delta_{cd}$ on M .

To prove this generalization of Lemma 5.21 we shall use the following generalization of Lemma 5.5.

6.6 Lemma. *Let M be an embedded submanifold of a smooth manifold V , and let m a point in M . Let $\{Z_1, \dots, Z_k\}$ be vector fields on V , defined in some neighborhood of $m \in V$, and assume that their values at m project to linearly independent vectors in the normal space $TV|_M / TM$. If a linear differential operator of the form*

$$D = \sum f_\alpha Z^\alpha$$

has the property that $(Df)(m) = 0$ for every smooth function f on V that vanishes on M , then all the coefficient functions f_α vanish at m . \square

Proof of Lemma 6.5. According to Lemma 6.6 the monomial operators X^α that use only normal vector fields in the local (G, H) -frame map by evaluation at m to a linearly independent set in $\text{Hom}(I_1(V, M), \mathbb{R})$. If we consider only monomial operators of H -order r or less, then this linearly independent set lies in the finite-dimensional vector space

$$\text{Hom}(I_1(V, M)/I_{r+1}(V, M), \mathbb{R}) \subseteq \text{Hom}(I_1(V, M), \mathbb{R})$$

and so, by linear algebra, associated to this finite linearly independent set in a finite-dimensional vector space there are functions $g_\beta \in I_1(V, M)$ with $Z^\alpha(g_\beta) = \delta_\alpha^\beta$ at the point m .

We want to adjust the functions g_β so that this relation holds near m in M , not only at the single point m . Let $h_{\alpha\beta} = Z^\alpha(g_\beta)$. This matrix of functions is the identity at m , and so is invertible near m . Let $h^{\alpha\beta}$ be the entries of the inverse matrix and define

$$f_\beta = \sum_\gamma h^{\beta\gamma} g_\gamma.$$

Then $Z^\alpha(f_\beta) = \delta_{\alpha\beta}$ on M , near m . Now, if we define z_c to be the function f_β associated to the vector field $Z^\beta = Z_c$, then the functions $\{z_c\}$ have the required properties. \square

Proof of Theorem 6.3. We shall use the vector fields and functions obtained above to show that the criteria in Lemma 2.4 are satisfied for every character φ of $A_0(V, M)$.

The degree zero part of $A_0(V, M)$ is $C^\infty(M)$, and φ restricts there to evaluation at some $m \in M$. Let $\{X_a\}$ be a local (G, H) -frame near m . Choose smooth functions $\{z_c\}$ on V as in Lemma 6.5. In addition, choose smooth functions $\{y_a\}$ on V , indexed by the members Y_a of the local (G, H) -frame that are tangent to M , so that

$$Y_a(y_b) = \delta_{ab} \quad \text{at } m \in V.$$

The functions $\{y_a, z_c\}$ are local coordinates in some open neighborhood U of $m \in V$, while the functions $\{y_a\}$ alone are local coordinates for $M \cap U$.

Now let Λ be the open set in $N_V M$ consisting of all those characters whose restriction to the degree zero part of $A_0(V, M)$ is evaluation at some point of $M \cap U$. It follows from Taylor's theorem that the elements

$$(6.1) \quad \langle y_a \rangle_0 \quad \text{and} \quad \langle z_c \rangle_{q_c}$$

smoothly generate $A_0(V, M)$ over Λ .

Moreover $A_0(U, M \cap U)$ is freely generated as an algebra over its degree zero part $C^\infty(M \cap U)$ by the classes $\langle z_c \rangle_{q_c}$. So if $\dim(M)=k$ and $\dim(V)=n$, then the map

$$\text{Spectrum}(A_0(V, M)) \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$$

given by evaluation on the generators (6.1) sends Λ homeomorphically to the open set $W \times \mathbb{R}^{n-k}$, where $W \subseteq \mathbb{R}^k$ is the range of the coordinates $\{y_c\}$ on $M \cap U$. \square

We shall now calculate the normal space $N_V M$ in terms of the osculating groups introduced in the last section. There is a natural map

$$(6.2) \quad N_V M \longrightarrow M$$

corresponding to the inclusion of $C^\infty(M)$ as the degree zero subalgebra of $A_0(V, M)$, and fiber of $N_V M$ over $m \in M$ identifies with the spectrum of the following algebra.

6.7 Definition. If $m \in M$, then we shall denote by $A_{0,m}(V, M)$ the quotient of $A_0(V, M)$ by the ideal in $A_0(V, M)$ generated by the vanishing ideal of m in $C^\infty(M)$. The formula

$$\varepsilon_m: \sum \langle a_q \rangle_q \longmapsto a_0(m)$$

defines a character of $A_{0,m}(V, M)$ that we shall call the *counit*.

6.8 Theorem. Let (M, G) be a filtered submanifold of a filtered manifold (V, H) and let m be a point in M . Let \mathcal{H}_m and \mathcal{G}_m be the osculating groups for $m \in V$ and $m \in M$, respectively. There is a unique H_m -equivariant algebra isomorphism

$$A_{0,m}(V, M) \longrightarrow A(\mathcal{H}_m/\mathcal{G}_m)$$

whose composition with evaluation at the identity coset in $\mathcal{H}_m/\mathcal{G}_m$ is the counit ε_m of $A_{0,m}(V, M)$.

6.9 Remark. Here $A(\mathcal{H}_m/\mathcal{G}_m)$ is the algebra of polynomial functions on the unipotent homogenous space $\mathcal{H}_m/\mathcal{G}_m$, or equivalently the algebra of polynomial functions on \mathcal{H}_v that are invariant under right translations by elements of \mathcal{G}_m .

Proof. The Lie algebra \mathfrak{h}_m acts on $A_{0,m}(V, M)$ by derivations according to the formula in Lemma 5.15, and this action exponentiates to a locally finite-dimensional action of H_v by automorphisms. The image of the orbit map

$$A_{0,m}(V, M) \longrightarrow A(\mathcal{H}_m)$$

associated to the counit ε_m is included in the right \mathcal{G}_m -invariant functions on $A(\mathcal{H}_m)$; this is a consequence of the fact that if $X \in \mathfrak{g}_m$, then

$$\varepsilon_m(\delta_X(a)) = 0$$

for every $a \in A_{0,m}(V, M)$. So we obtain an orbit homomorphism

$$A_{0,m}(V, M) \longrightarrow A(\mathcal{H}_m/\mathcal{G}_m),$$

and it remains to show that it is an isomorphism. We shall use a variation on the argument used to prove Theorem 5.19.

Filter $A_{0,m}(V, M)$ by order of vanishing of functions in the ordinary sense at m . Using the coordinates of the previous lemma, the associated graded algebra is freely generated by the classes $\langle z_c \rangle_{q_c}$.

Filter $A(\mathcal{H}_m/\mathcal{G}_m)$ by order of vanishing in the ordinary sense at the basepoint in $\mathcal{H}_m/\mathcal{G}_m$. The associated graded algebra is freely generated by the normal dual vectors $\langle Z_c \rangle^* \in (\mathfrak{h}_m/\mathfrak{g}_m)^*$.

Our orbit map is filtration preserving, we find that it induces an isomorphism on associated graded algebras; indeed it maps $\langle z_c \rangle_{q_c}$ to $\langle Z_c \rangle^*$. \square

6.10 Remark. The algebra $A_0(V, M)$ consists of those smooth functions on the normal space $N_V M$ whose restrictions to all of the fibers of (6.2) are polynomial functions.

7 Deformation Spaces for Filtered Manifolds

In this section we shall construct the deformation space associated to a filtered submanifold of a filtered manifold. We shall copy Section 3 almost verbatim.

7.1 Definition. Let (M, G) be a filtered submanifold of a filtered manifold (V, H) . Denote by $A(V, M)$ the algebra of Laurent polynomials

$$\sum_{n \in \mathbb{Z}} a_n t^{-n}$$

whose coefficients are smooth, real-valued functions on V that satisfy the condition

$$q > 0 \quad \Rightarrow \quad a_q \text{ vanishes to H-order } q \text{ on } M.$$

The deformation space $\mathbb{N}_V M$ is the spectrum of $A(V, M)$.

As is the case for ordinary manifolds, the deformation space is a union

$$\mathbb{N}_V M = N_V M \times \{0\} \sqcup \bigsqcup_{\lambda \in \mathbb{R}^\times} V \times \{\lambda\},$$

(but of course with the normal space from the previous section).

7.2 Theorem. *The deformation space $\mathbb{N}_V M$ is a smooth manifold in such a way that the sheaf of smooth functions is the sheaf \mathcal{S} from Definition 2.2.*

Proof. We shall follow the proof of Theorem 3.8, and we shall use the same coordinate functions $\{y_a\}$ and $\{z_c\}$ as in the proof of Theorem 6.3, defined in a neighborhood U of $m \in V$. Let $\Lambda \subseteq \mathbb{N}_V M$ be the open subset consisting of all (u, λ) with $u \in U$ and $\lambda \neq 0$, together with all the elements $(X_m, 0)$, with $X_m \in \mathcal{H}_m/G_m$. The elements

$$(7.1) \quad t, \quad y_a, \quad \text{and} \quad z_c t^{-q_c}$$

of $A(V, M)$ satisfy the conditions of Lemma 2.4. If $W \subseteq \mathbb{R}^n$ is the image of the coordinates $\{y_a, z_c\}$, then the functions (7.1) map Λ homeomorphically to the open subset

$$\{ (\lambda, \{y_a\}, \{z_c\}) : (\{y_a\}, \{\lambda^{q_c} z_c\}) \in W \}$$

of \mathbb{R}^{n+1} . □

7.3 Remark. The deformation space was previously constructed by Van Erp [Erp05] and Ponge [Pon06] in the 2-step case, and then by Choi and Ponge [CP15], and also by Van Erp and Yuncken [EY16], in the general case. These authors considered the diagonal embedding of M in $M \times M$, which is the most important case, obtaining an analogue for filtered manifolds of Connes's *tangent groupoid* [Con94, Chapter 2, Section 5]. Connes gave a proof of the Atiyah-Singer theorem using the tangent groupoid; see [Erp10a] for a proof of an index theorem for contact manifolds using a similar approach.

8 Euler-Like Vector Fields on Filtered Manifolds

Let (M, G) be a filtered submanifold of a filtered manifold (V, H) , and form the normal space $N_V M$.

8.1 Definition. Let (M, G) be a filtered submanifold of a filtered manifold (V, H) . An *Euler-like vector field* for the embedding of M into V is a vector field E with the property that if f is a smooth function on V that vanishes on M to H -order q , then

$$E(f) = q \cdot f + r$$

where r is a smooth function that vanishes on M to H -order $q+1$ or higher.

8.2 Example. If $m \in M$ and if $\{y_a, z_c\}$ is the local coordinate system defined near $m \in V$, that was used in the proofs of Theorems 6.3 and 7.2, then formula

$$E = \sum_c q_c \cdot z_c \cdot \frac{\partial}{\partial z_c}$$

defines an Euler-like vector field near m . A global Euler-like vector field can be assembled from locally defined Euler-like vector fields of this type using a partition of unity.

Our aim is to relate Euler-like vector fields to tubular neighborhood embeddings, as in Theorem 1.2. An interesting feature of the filtered manifold case that we are now considering is that it is not immediately clear what the appropriate notion of tubular neighborhood embedding should be (for instance, the normal space $N_V M$ is not itself a filtered manifold, so we cannot insist that tubular neighborhood embeddings be isomorphisms of filtered manifolds). So we shall let the analogue of Theorem 1.2 determine the definition of tubular neighborhood embedding.

To define the appropriate notion of tubular neighbourhood embedding we shall need to define a “zero section” of the normal space, and then examine the vertical tangent bundle for the submersion

$$N_V M \longrightarrow M$$

at the zero section. First, the homomorphism

$$\begin{aligned} A_0(V, M) &\longrightarrow C^\infty(M) \\ \sum \langle a_q \rangle_q &\longmapsto \langle a_0 \rangle_0 \end{aligned}$$

defines an inclusion of M into $N_V M$ that will be our zero section. Next, the vertical tangent space at a point m in the zero section identifies with the quotient of Lie algebras $\mathfrak{h}_m / \mathfrak{g}_m$. Each of \mathfrak{h}_m and \mathfrak{g}_m is a graded Lie algebra, and we shall write

$$\mathfrak{h}_m^q = H_m^q / H_m^{q-1} \quad \text{and} \quad \mathfrak{g}_m^q = G_m^q / G_m^{q-1}.$$

8.3 Definition. Let (M, G) be a filtered submanifold of a filtered manifold (V, H) . A *tubular neighborhood embedding* of $N_V M$ into V is a diffeomorphism from a neighborhood of $M \subseteq N_V M$ to a neighborhood of $M \subseteq V$ with the following properties:

- (a) The diffeomorphism is the identity on M
- (b) At each point of M the differential maps the vertical space $\mathfrak{h}_m^q/\mathfrak{g}_m^q$ into H_m^q , and the composition

$$\mathfrak{h}_m^q/\mathfrak{g}_m^q \longrightarrow H_m^q \longrightarrow \mathfrak{h}_m^q/\mathfrak{g}_m^q$$

with the natural projection is the identity.

The normal space $N_V M$ carries a natural vector field, which we shall call the Euler vector field, as follows:

8.4 Definition. The *Euler vector field* on $N_V M$ is the vector field associated to the smooth derivation of $A_0(V, M)$ given by

$$\sum_q \langle a_q \rangle_q \longmapsto \sum_q q \cdot \langle a_q \rangle_q.$$

8.5 Remark. The normal space $N_V M$ is not naturally a filtered manifold, in general. But if M is a point, then $N_V M$ is simply the unipotent group \mathcal{H}_V , and this is a filtered manifold. In this case, the Euler vector field is Euler-like in the sense of Definition 8.1.

The Euler vector field generates a flow $\{\rho_s\}$ on $N_V M$ that is easy to describe in group-theoretic terms. First, there is a one-parameter group of Lie algebra automorphisms of the graded Lie algebra

$$\mathfrak{h}_m = \bigoplus_{q=1}^r H_m^q/H_m^{q-1}$$

that multiplies the degree q summand by e^{tq} . This one-parameter group exponentiates to a one-parameter group of automorphisms of the unipotent group \mathcal{H}_m that maps the subgroup \mathcal{G}_m to itself, and therefore induces a flow $\{\rho_s\}$ on the homogeneous space $\mathcal{H}_m/\mathcal{G}_m$, as required.

8.6 Definition. Denote by \mathbb{A} the vector field on $N_V M$ that generates the flow

$$\alpha_s: \begin{cases} (v, \lambda) \longmapsto (v, e^s \lambda) \\ (X, 0) \longmapsto (\rho_{-s} X, 0) \end{cases}$$

8.7 Lemma. *If E is an Euler-like vector field for the inclusion of M into V , then the vector field*

$$T = \lambda^{-1}E + \frac{\partial}{\partial \lambda}$$

on the open subset $V \times \mathbb{R}^\times \subseteq \mathbb{N}_V M$ extends to a vector field on $\mathbb{N}_V M$ with

$$\lambda \cdot T = A + E. \quad \square$$

Repeating the argument from Section 4 we find that:

8.8 Theorem. *Let (M, G) be a filtered submanifold of a filtered manifold (V, H) . The correspondence that associates to each tubular neighborhood embedding the associated Euler-like vector field on V is bijection from germs of tubular neighborhood embeddings to germs of Euler-like vector fields.* \square

8.9 Remark. In the case where M is a point, the inverse

$$V \longrightarrow \mathcal{H}_m$$

of the tubular neighborhood embedding corresponds to a system of Carnot coordinates, as in [CP15, Section 7] and Remark 5.23.

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